Decay estimates for steady solutions of the Navier-Stokes equations in two dimensions in the presence of a wall

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January 18, 2013

Abstract

Let ω be the vorticity of a stationary solution of the two-dimensional Navier-Stokes equations with a drift term parallel to the boundary in the half-plane $\Omega_+ = \left\{ (x,y) \in \mathbb{R}^2 \middle| \ y > 1 \right\}$, with zero Dirichlet boundary conditions at $y = 1$ and at infinity, and with a small force term of compact support. Then, $|xy\omega(x,y)|$ is uniformly bounded in Ω_+. The proof is given in a specially adapted functional framework and the result is a key ingredient for obtaining information on the asymptotic behavior of the velocity at infinity.

Keywords: Navier-Stokes; exterior domain; fluid-structure interaction; asymptotic behavior

Contents

*Supported by the Swiss National Science Foundation (Grant No. 200021-124403).
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‡Supported by the Swiss National Science Foundation (Grant No. 200021-124403).

1 Introduction

In this paper we consider the steady Navier-Stokes equations in a half-plane $\Omega_+ = \left\{ (x,y) \in \mathbb{R}^2 \middle| \ y > 1 \right\}$ with a drift term parallel to the boundary, a small driving force of compact support, with zero Dirichlet boundary conditions at the boundary of the half plane and at infinity. See [12] and [13] for a detailed motivation of this problem. Existence of a strong solution for this system was proved in [12] together with a basic bound on the decay at infinity, and the existence of weak solutions was shown in [13]. By elliptic regularity weak solutions are smooth, and their only possible shortcoming is the behavior at infinity, since the boundary condition may not be satisfied there in a pointwise sense. In [13] it was also shown that for small forces there is only one weak solution. This unique weak solution therefore coincides with the strong solution and satisfies as a consequence the boundary condition at infinity in a pointwise sense.

The aim of this paper is to provide additional information concerning the behavior of this solution at infinity by analyzing the solution obtained in [12] in a more stringent functional setting. More precisely, we obtain more information on the decay behavior of the vorticity of the flow. Bounds on vorticity as a step towards bounds on the velocity are a classical procedure in asymptotic analysis of fluid flows (see the seminal papers [6], [7] and [1]). In [12] and the current work, the equation for the vorticity is Fourier-transformed with respect to the coordinate x parallel to the wall, and then rewritten as a dynamical system with the coordinate y perpendicular to the wall playing the role of time. In this setting information on the behavior of the vorticity at infinity is studied by analyzing the Fourier transform at $k = 0$, with k the Fourier conjugate variable of x. In the present work, we also control the derivative of the Fourier transform of the vorticity, which yields more precise decay estimates for the vorticity and the velocity field in direct space than the ones found in [12]. Our proof is then based on a new linear fixed point problem involving the solution obtained in [12] and the derivative of the vorticity with respect to k.

Since the original equation is elliptic, the dynamical system under consideration contains stable and unstable modes and no spectral gap, so that standard versions of the center manifold theorem are not sufficient to prove existence of solutions. Functional techniques that allow to deal with such a situation go back to [5] and were adapted to the case of the Navier-Stokes equations in [14] and in [15], [16]. For a general review see [10]. The linearized version of the current problem was studied in [11]. A related problem in three dimensions was discussed in [8].

The results of the present paper are the basis for the work described in [2], where we extract several orders of an asymptotic expansion of the vorticity and the velocity field at infinity. The asymptotic velocity field obtained this way is divergence-free and may be used to define artificial boundary conditions of Dirichlet type when the system of equations is restricted to a finite sub-domain to be solved numerically. The use of asymptotic terms as artificial boundary conditions was pioneered in [3] for the related problem of an exterior flow in the whole space in two dimensions, and in [9] for the case in three dimensions.

Let $\mathbf{x} = (x, y)$, and let $\Omega_+ = \left\{ (x,y) \in \mathbb{R}^2 \middle| \ y > 1 \right\}$. The model under consideration is given by the Navier-Stokes equations with a drift term parallel to the boundary,

$$-\partial_x \boldsymbol{u} + \Delta \boldsymbol{u} = \boldsymbol{F} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p \ , \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \ , \tag{2}$$

subject to the boundary conditions

$$\boldsymbol{u}(x, 1) = 0 \ , \qquad x \in \mathbb{R} \ , \tag{3}$$

$$\lim_{\mathbf{x} \to \infty} \boldsymbol{u}(\mathbf{x}) = 0 \ . \tag{4}$$

The following theorem is our main result.

Theorem 1 *For all $\boldsymbol{F} \in C_c^\infty(\Omega_+)$ with $\boldsymbol{F}$ sufficiently small in a sense to be defined below, there exist a unique vector field $\boldsymbol{u} = (u, v)$ and a function p satisfying the Navier-Stokes equations (1), (2) in Ω_+ subject to the boundary conditions (3) and (4). Moreover, there exists a constant $C > 0$, such that $|y^{3/2} u(x,y)| + |y^{3/2} v(x,y)| + |y^3 \omega(x,y)| + |xy\omega(x,y)| \leq C$, for all $(x,y) \in \Omega_+$.*

This theorem is a consequence of Theorem 9 which is proved in Section 5. The crucial improvement with respect to [12] is the bound on the function $xy\omega(x,y)$.

The paper is organized as follows. In Section 2 we rewrite (1) and (2) as a dynamical system with y playing the role of time, and Fourier-transform the equations with respect to the variable x. Then, in Section 3, we recall the integral equations for the vorticity discussed in [12] and complement them by the ones for the derivative with respect to k. We then introduce in Section 4 certain well adapted Banach spaces which encode the information concerning the decay of the functions at infinity. Finally, in Section 5, we reformulate the problem of showing the existence of the derivative of vorticity with respect to k as the fixed point of a continuous map, based on the existence of solutions proved in [12]. We present in Sections 6 and 7 the proofs of the lemmas used in Section 5. In the appendix, we recall results from [12] which are needed here.

2 Reduction to an evolution equation

We recall the procedure used in [12] to frame the Navier-Stokes equations for the studied case as a dynamical system. Let $\boldsymbol{u} = (u, v)$ and $\boldsymbol{F} = (F_1, F_2)$. Then, equations (1) and (2) are equivalent to

$$\omega = -\partial_y u + \partial_x v \ , \tag{5}$$

$$-\partial_x \omega + \Delta\omega = \partial_x(u\omega) + \partial_y(v\omega) + \partial_x F_2 - \partial_y F_1 \ , \tag{6}$$

$$\partial_x u + \partial_y v = 0 \ . \tag{7}$$

The function ω is the vorticity of the fluid. Once equations (5)-(7) are solved, the pressure p can be obtained by solving the equation

$$\Delta p = -\nabla \cdot (\boldsymbol{F} + \boldsymbol{u} \cdot \nabla \boldsymbol{u})$$

in Ω_+, subject to the Neumann boundary condition

$$\partial_y p(x,1) = \partial_y^2 v(x,1) \ .$$

Let

$$q_0 = u\omega \ , \tag{8}$$

$$q_1 = v\omega \ , \tag{9}$$

and let furthermore

$$Q_0 = q_0 + F_2 \ , \tag{10}$$

$$Q_1 = q_1 - F_1 \ . \tag{11}$$

We then rewrite the second order differential equation (6) as a first order system

$$\partial_y \omega = \partial_x \eta + Q_1 \ , \tag{12}$$

$$\partial_y \eta = -\partial_x \omega + \omega + Q_0 \ . \tag{13}$$

Note that, unlike the right-hand side of (6), the expressions for Q_0 and Q_1 do not contain derivatives. This is due to the fact that, in contrast to standard practice, we did not set, say, $\partial_y \omega = \eta$, but we chose with (12) a more sophisticated definition. The fact that the nonlinear terms in (12), (13) do not contain derivatives simplifies the analysis of the equations considerably. An additional trick allows to reduce complexity even further. Namely, we can replace (7) and (5) with the equations

$$\partial_y \psi = -\partial_x \varphi - Q_1 \ , \tag{14}$$

$$\partial_y \varphi = \partial_x \psi + Q_0 \ , \tag{15}$$

if we use the decomposition

$$u = -\eta + \varphi \ , \tag{16}$$
$$v = \omega + \psi \ . \tag{17}$$

The point is that in contrast to u and v the functions ψ and φ decouple on the linear level from ω and η. Since, on the linear level we have $\Delta\varphi = 0$ and $\Delta\psi = 0$, it will turn out that φ and ψ have a dominant asymptotic behavior which is harmonic when Q_0 and Q_1 are small.

Equations (12)-(15) are a dynamical system with y playing the role of time. We now take the Fourier transform in the x-direction.

Definition 2 *Let $\hat{f}$ be a complex valued function on Ω_+. Then, we define the inverse Fourier transform $f = \mathcal{F}^{-1}[\hat{f}]$ by the equation,*

$$f(x,y) = \mathcal{F}^{-1}[\hat{f}](x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ikx}\hat{f}(k,y)dk \ ,$$

and $\hat{h} = \hat{f} * \hat{g}$ by

$$\hat{h}(k,y) = (\hat{f} * \hat{g})(k,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k-k',y)\hat{g}(k',y)dk' \ ,$$

whenever the integrals make sense. We note that for a function f which is smooth and of compact support in Ω_+ we have $f = \mathcal{F}^{-1}[\hat{f}]$, where

$$\hat{f}(k,y) = \mathcal{F}[f](k,y) = \int_{\mathbb{R}} e^{ikx} f(x,y)dx \ ,$$

and that $fg = \mathcal{F}^{-1}[\hat{f} * \hat{g}]$.

With these definitions we have in Fourier space, instead of (12)-(15), the equations

$$\partial_y\hat{\omega} = -ik\hat{\eta} + \hat{Q}_1 \ , \tag{18}$$
$$\partial_y\hat{\eta} = (ik+1)\hat{\omega} + \hat{Q}_0 \ , \tag{19}$$
$$\partial_y\hat{\psi} = ik\hat{\varphi} - \hat{Q}_1 \ , \tag{20}$$
$$\partial_y\hat{\varphi} = -ik\hat{\psi} + \hat{Q}_0 \ . \tag{21}$$

From (10) and (11) we get

$$\hat{Q}_0 = \hat{q}_0 + \hat{F}_2 \ , \tag{22}$$
$$\hat{Q}_1 = \hat{q}_1 - \hat{F}_1 \ , \tag{23}$$

from (8) and (9) we get

$$\hat{q}_0 = \hat{u} * \hat{\omega} \ , \tag{24}$$
$$\hat{q}_1 = \hat{v} * \hat{\omega} \ , \tag{25}$$

and instead of (16) and (17) we have the equations

$$\hat{u} = -\hat{\eta} + \hat{\varphi} \ , \tag{26}$$
$$\hat{v} = \hat{\omega} + \hat{\psi} \ . \tag{27}$$

3 Integral equations

We now reformulate the problem of finding a solution to (18)-(21) which satisfies the boundary conditions (3) and (4) in terms of a system of integral equations. The equations for $\hat{\omega}$, $\hat{\eta}$, $\hat{\varphi}$ and $\hat{\psi}$ are as in [12]. In particular we recall that

$$\hat{\omega} = \sum_{m=0}^{1} \sum_{n=1}^{3} \hat{\omega}_{n,m} \ , \tag{28}$$

where, for $n = 1, 2, 3$, $m = 0, 1$,

$$\hat{\omega}_{n,m}(k,t) = \check{K}_n(k,t-1) \int_{I_n} \check{f}_{n,m}(k,s-1) \hat{Q}_m(k,s) ds \ , \tag{29}$$

where, for $k \in \mathbb{R} \setminus \{0\}$ and σ, $\tau \geq 0$,

$$\check{K}_n(k,\tau) = \frac{1}{2} e^{-\kappa\tau} \ , \text{ for } n = 1, 2 \ , \tag{30}$$

$$\check{K}_3(k,\tau) = \frac{1}{2}(e^{\kappa\tau} - e^{-\kappa\tau}) \ , \tag{31}$$

and

$$\check{f}_{1,0}(k,\sigma) = \frac{ik}{\kappa} e^{\kappa\sigma} - \frac{(|k|+\kappa)^2}{\kappa} e^{-\kappa\sigma} + 2\left(|k|+\kappa\right) e^{-|k|\sigma} \ , \tag{32}$$

$$\check{f}_{2,0}(k,\sigma) = 2\left(\kappa + |k|\right) \left(e^{-|k|\sigma} - e^{-\kappa\sigma}\right) \ , \tag{33}$$

$$\check{f}_{3,0}(k,\sigma) = \frac{ik}{\kappa} e^{-\kappa\sigma} \ , \tag{34}$$

$$\check{f}_{1,1}(k,\sigma) = e^{\kappa\sigma} + \frac{(|k|+\kappa)^2}{ik} e^{-\kappa\sigma} - 2\frac{|k|\left(|k|+\kappa\right)}{ik} e^{-|k|\sigma} \ , \tag{35}$$

$$\check{f}_{2,1}(k,\sigma) = 2\left(\frac{|k|\left(|k|+\kappa\right)}{ik} - 1\right) e^{-\kappa\sigma} - 2\frac{|k|\left(|k|+\kappa\right)}{ik} e^{-|k|\sigma} \ , \tag{36}$$

$$\check{f}_{3,1}(k,\sigma) = -e^{-\kappa\sigma} \ , \tag{37}$$

and where $I_1 = [1,t]$ and $I_2 = I_3 = [t,\infty)$.

We introduce the integral equation for $\partial_k \hat{\omega}$, noting that $\hat{\omega}$ is continuous at $k = 0$ (see [12]). From (29) we get that

$$\partial_k \hat{\omega} = \sum_{m=0}^{1} \sum_{n=1}^{3} \sum_{l=1}^{3} \partial_k \hat{\omega}_{l,n,m} \ , \tag{38}$$

where, for $n = 1, 2, 3$, $m = 0, 1$,

$$\partial_k \hat{\omega}_{1,n,m}(k,t) = \partial_k \check{K}_n(k,t-1) \int_{I_n} \check{f}_{n,m}(k,s-1) \hat{Q}_m(k,s) ds \ , \tag{39}$$

$$\partial_k \hat{\omega}_{2,n,m}(k,t) = \check{K}_n(k,t-1) \int_{I_n} \partial_k \check{f}_{n,m}(k,s-1) \hat{Q}_m(k,s) ds \ , \tag{40}$$

$$\partial_k \hat{\omega}_{3,n,m}(k,t) = \check{K}_n(k,t-1) \int_{I_n} \check{f}_{n,m}(k,s-1) \partial_k \hat{Q}_m(k,s) ds \ , \tag{41}$$

where, for $k \in \mathbb{R} \setminus \{0\}$ and σ, $\tau \geq 0$,

$$\partial_k \check{K}_n(k,\tau) = \frac{1}{4} \frac{2k - i}{\kappa} e^{-\kappa\tau} \ , \text{ for } n = 1, 2 \ , \tag{42}$$

$$\partial_k \check{K}_3(k,\tau) = \frac{1}{4} \frac{2k - i}{\kappa} (e^{\kappa\tau} + e^{-\kappa\tau}) \ , \tag{43}$$

where $\check{f}_{n,m}$ is as above, where

$$\partial_k \check{f}_{1,0}(k,\sigma) = \frac{i}{2\kappa}(e^{\kappa\sigma} + e^{-\kappa\sigma} - 2e^{-|k|\sigma}) - \frac{ik^2}{2\kappa^3}(e^{\kappa\sigma} - e^{-\kappa\sigma}) + \frac{2}{\kappa}\frac{k^2+|k|\kappa}{k}(e^{-|k|\sigma} - e^{-\kappa\sigma}) + i\frac{k^2+\kappa^2}{2\kappa^2}(e^{\kappa\sigma} - e^{-\kappa\sigma})\sigma + \frac{k^2+|k|\kappa}{k}\frac{k^2+\kappa^2}{\kappa^2}e^{-\kappa\sigma}\sigma - 2\frac{k^2+|k|\kappa}{k}e^{-|k|\sigma}\sigma \ , \tag{44}$$

$$\partial_k \check{f}_{2,0}(k,\sigma) = \frac{(|k|+\kappa)^2}{\kappa k}(e^{-|k|\sigma} - e^{-\kappa\sigma}) - 2\frac{\kappa+|k|}{\kappa k}\left(|k|\kappa e^{-|k|\sigma} - \frac{k^2+\kappa^2}{2}e^{-\kappa\sigma}\right)\sigma \ , \tag{45}$$

$$\partial_k \check{f}_{3,0}(k,\sigma) = \frac{k}{2\kappa^3}e^{-\kappa\sigma} - i\frac{k^2+\kappa^2}{2\kappa^2}\sigma e^{-\kappa\sigma} \ , \tag{46}$$

$$\partial_k \check{f}_{1,1}(k,\sigma) = i\frac{(|k|+\kappa)^2}{\kappa|k|}(e^{-|k|\sigma} - e^{-\kappa\sigma}) + \frac{k^2+\kappa^2}{2\kappa k}(e^{\kappa\sigma} + e^{-\kappa\sigma})\sigma + 2i\frac{k^2+|k|\kappa}{k^2}\left(\frac{k^2+\kappa^2}{2\kappa}e^{-\kappa\sigma} - |k|e^{-|k|\sigma}\right)\sigma \ , \tag{47}$$

$$\partial_k \check{f}_{2,1}(k,\sigma) = i\frac{(|k|+\kappa)^2}{\kappa|k|}(e^{-\kappa\sigma} - e^{-|k|\sigma}) + i(|k|+\kappa)\frac{k^2+\kappa^2}{k^2}e^{-\kappa\sigma}\sigma - 2i\left(|k|+\kappa\right)e^{-|k|\sigma}\sigma \ , \tag{48}$$

$$\partial_k \check{f}_{3,1}(k,\sigma) = \frac{k^2+\kappa^2}{2\kappa k}e^{-\kappa\sigma}\sigma \ . \tag{49}$$

and where the functions

$$\partial_k \hat{Q}_0 = \partial_k \hat{q}_0 + \partial_k \hat{F}_2 \ ,$$
$$\partial_k \hat{Q}_1 = \partial_k \hat{q}_1 - \partial_k \hat{F}_1 \ ,$$

are obtained from (22) and (23). Since $\hat{q}_0$ and $\hat{q}_1$ are convolution products (see (24) and (25)), and noting that $\hat{u}$ and $\hat{v}$ are continuous bounded functions on $\mathbb{R}$, that $\hat{\omega}$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}\setminus\{0\}$ and that $\partial_k\hat{\omega}$ is absolutely integrable, we conclude (see [4, Proposition 8.8, page 241]) that $\hat{q}_0$ and $\hat{q}_1$ are continosly differentiable functions and that

$$\partial_k \hat{q}_0 = \hat{u} * \partial_k \hat{\omega} \ , \tag{50}$$
$$\partial_k \hat{q}_1 = \hat{v} * \partial_k \hat{\omega} \ . \tag{51}$$

This means that it is sufficient to add equation (38) to the ones for $\hat{\omega}$, $\hat{\eta}$, $\hat{\varphi}$ and $\hat{\psi}$ in order to get a set of integrals equations determining also $\partial_k\hat{\omega}$.

Remark 3 *The products $\check{K}_n\check{f}_{n,m}$ are equal to $K_n f_{n,m}$ as defined in [12], and we have $\check{K}_{n=1,2} = K_{n=1,2}$, $\check{K}_3 = \frac{ik}{\kappa}K_3$, $\check{f}_{n=1,2;m} = f_{n=1,2;m}$ and $\check{f}_{3,m} = \frac{\kappa}{ik}f_{3,m}$. We chose to rewrite the equations in the new form for convenience later on.*

4 Functional framework

We recall the definition of the function spaces introduced in [12] and extend it to include functions with a certain type of singular behavior. Let α, $r \geq 0$, $k \in \mathbb{R}$, $t \geq 1$, and let

$$\mu_{\alpha,r}(k,t) = \frac{1}{1 + (|k|t^r)^\alpha} \ . \tag{52}$$

Let furthermore

$$\bar{\mu}_\alpha = \mu_{\alpha,1}(k,t) \ ,$$
$$\tilde{\mu}_\alpha = \mu_{\alpha,2}(k,t) \ .$$

We also define

$$\kappa = \sqrt{k^2 - ik} \ , \tag{53}$$

and

$$\Lambda_- = -\operatorname{Re}(\kappa) = -\frac{1}{2}\sqrt{2\sqrt{k^2+k^4}+2k^2} \ . \tag{54}$$

Throughout this paper we use the inequalities

$$|\kappa| = (k^2+k^4)^{1/4} \leq |k|^{1/2} + |k| \leq 2^{3/4}|\kappa| \leq 2^{3/4}(1+|k|) \ . \tag{55}$$

We have in particular that

$$|k|^{\frac{1}{2}} \leq \text{const.}|\kappa| \ , \tag{56}$$

and that

$$e^{\Lambda_-\sigma} \leq e^{-|k|\sigma} \ , \tag{57}$$

which will play a crucial role for small and large values of k, respectively.

Definition 4 *We define, for fixed $\alpha \geq 0$, and n, p, $q \geq 0$, $\mathcal{B}^n_{\alpha,p,q}$ to be the Banach space of functions $\hat{f}\colon \mathbb{R}\setminus\{0\}\times[1,\infty)\to\mathbb{C}$, for which $\hat{f}_n = \kappa^n \cdot \hat{f} \in C(\mathbb{R}\setminus\{0\}\times[1,\infty),\mathbb{C})$, and for which the norm*

$$\left\|\hat{f};\mathcal{B}^n_{\alpha,p,q}\right\| = \sup_{t\geq 1}\sup_{k\in\mathbb{R}\setminus\{0\}} \frac{|\hat{f}_n(k,t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(k,t)+\frac{1}{t^q}\tilde{\mu}_\alpha(k,t)}$$

is finite. We use the shorthand $\mathcal{B}_{\alpha,p,q}$ for $\mathcal{B}^0_{\alpha,p,q}$. Furthermore we set, for $\alpha > 2$,

$$\mathcal{D}^1_{\alpha-1,p,q} = \mathcal{B}^1_{\alpha,p,q}\times\mathcal{B}^1_{\alpha-\frac{1}{2},p+\frac{1}{2},q+\frac{1}{2}}\times\mathcal{B}^1_{\alpha-1,p+\frac{1}{2},q+1} \ ,$$
$$\mathcal{V}_\alpha = \mathcal{B}_{\alpha,\frac{5}{2},1}\times\mathcal{B}_{\alpha,\frac{1}{2},0}\times\mathcal{B}_{\alpha,\frac{1}{2},1} \ .$$

Remark 5 *We present two elementary properties of the spaces $\mathcal{B}^n_{\alpha,p,q}$, which will be routinely used without mention. Let α, $\alpha' \geq 0$, and p, p', q, $q' \geq 0$, then*

$$\mathcal{B}^n_{\alpha,p,q}\cap\mathcal{B}^n_{\alpha',p',q'} \subset \mathcal{B}^n_{\min\{\alpha',\alpha,\},\min\{p',p\},\min\{q',q\}} \ .$$

In addition we have

$$\mathcal{B}^n_{\alpha,p,q} \subset \mathcal{B}^n_{\alpha,\min\{p,q\},\infty} \ ,$$

where the space with $q=\infty$ is to be understood to contain functions for which the norm

$$\left\|\hat{f};\mathcal{B}^n_{\alpha,p,\infty}\right\| = \sup_{t\geq 1}\sup_{k\in\mathbb{R}\setminus\{0\}} \frac{|\hat{f}_n(k,t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(k,t)}$$

is finite.

5 Existence of solutions

In [12] it was shown that one can rewrite the integral equations as a fixed point problem, and that, for $\boldsymbol{F}$ sufficiently small, there exist functions $\hat{\omega}$, $\hat{u}$ and $\hat{v}$, that are solution to (5)-(7), satisfying the boundary conditions (3) and (4). More precisely, we have, for $\alpha > 3$,

$$\hat{\omega} \in \mathcal{B}_{\alpha,\frac{5}{2},1} \ , \tag{58}$$
$$\hat{u} \in \mathcal{B}_{\alpha,\frac{1}{2},0} \ , \tag{59}$$
$$\hat{v} \in \mathcal{B}_{\alpha,\frac{1}{2},1} \ , \tag{60}$$

and, for $i = 0, 1$,

$$\hat{Q}_i \in \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \ . \tag{61}$$

We now show that using this solution as a starting point, we may define a linear fixed point problem with a unique solution for $\partial_k \hat{\omega}$. The structure of (38) is rather complicated and it turns out to be necessary to decompose the sum into three parts which are analyzed independently. Let $\hat{\mathbf{d}} = (\hat{d}_1, \hat{d}_2, \hat{d}_3)$ where

$$\hat{d}_l = \sum_{m=0}^{1} \sum_{n=1}^{3} \partial_k \hat{\omega}_{l,n,m} \ ,$$

then $\partial_k \hat{\omega} = \sum_{l=1}^{3} \hat{d}_l$. The function $\hat{d}_3$ depends on $\partial_k \hat{\omega}$, but $\hat{d}_1$ and $\hat{d}_2$ do not.

Proposition 6 *The functions $\hat{d}_1$ and $\hat{d}_2$ are in $\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$.*

Proof. See Sections 7.1 and 7.2. ■

We now define the fixed point problem.

Lemma 7 *Let $\alpha > 3$, and let $\hat{u}$ and $\hat{v}$ be as in (59) and (60) respectively. Then,*

$$\begin{array}{cccc} \mathfrak{L}_1 : & \mathcal{D}^1_{\alpha-1,\frac{3}{2},0} & \to & \mathcal{B}_{\alpha,\frac{3}{2},1} \times \mathcal{B}_{\alpha,\frac{3}{2},2} \\ & \hat{d} & \longmapsto & \begin{pmatrix} \hat{u} * \hat{d} \\ \hat{v} * \hat{d} \end{pmatrix} \ , \end{array}$$

defines a continuous linear map.

Proof. The map $\mathfrak{L}_1$ is linear by definition of the convolution operation. Using Corollary 12 we get that the map $\mathfrak{L}_1$ is bounded, since

$$\left\| \hat{u} * \hat{d}; \mathcal{B}_{\alpha,\frac{3}{2},1} \right\| \leq \text{const.} \left\| \hat{u}; \mathcal{B}_{\alpha,\frac{1}{2},0} \right\| \cdot \left\| d; \mathcal{D}^1_{\alpha-1,\frac{3}{2},0} \right\| \ , \tag{62}$$

and

$$\left\| \hat{v} * \hat{d}; \mathcal{B}_{\alpha,\frac{3}{2},2} \right\| \leq \text{const.} \left\| \hat{v}; \mathcal{B}_{\alpha,\frac{1}{2},1} \right\| \cdot \left\| d; \mathcal{D}^1_{\alpha-1,\frac{3}{2},0} \right\| \ . \tag{63}$$

■

Lemma 8 *Let $\alpha > 3$, $\hat{d}_3 = \sum_{m=0}^{1} \sum_{n=1}^{3} \partial_k \hat{\omega}_{3,n,m}$ and let $\partial_k \hat{\omega}_{3,n,m}$ be given by (41). Then, we have*

$$\begin{array}{cccc} \mathfrak{L}_2 : & \mathcal{B}_{\alpha,\frac{3}{2},1} \times \mathcal{B}_{\alpha,\frac{3}{2},2} & \to & \mathcal{D}^1_{\alpha-1,\frac{3}{2},0} \\ & \begin{pmatrix} \partial_k \hat{Q}_0 \\ \partial_k \hat{Q}_1 \end{pmatrix} & \longmapsto & \hat{d}_3 \ , \end{array}$$

which defines a continuous linear map.

Proof. The map $\mathfrak{L}_2$ is linear by definition of $\hat{d}_3$ and is proved to be bounded in Section 7.3. ■

5.1 Proof of Theorem 1

Theorem 1 is a consequence of the following theorem.

Theorem 9 (Existence) *Let $\alpha > 3$, $\boldsymbol{F} = (F_1, F_2) \in C_c^\infty(\Omega_+)$, and let $\hat{\boldsymbol{F}} = (\hat{F}_1, \hat{F}_2)$ be the Fourier transform of $\boldsymbol{F}$. If $\|(\hat{F}_2, -\hat{F}_1); \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \times \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}\|$ is sufficiently small, then there exists a unique solution $(\hat{\omega}, \hat{u}, \hat{v}, \hat{\mathbf{d}})$ in $\mathcal{V}_\alpha \times \mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$.*

Proof. We have the existence and uniqueness of $(\hat{\omega}, \hat{u}, \hat{v}) \in \mathcal{V}_\alpha$ thanks to [12] and [13]. Since $\alpha > 3$, we have by Lemmas 7 and 8 that the map $\mathfrak{C} : \mathcal{D}^1_{\alpha-1,\frac{3}{2},0} \to \mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, $x \mapsto \mathfrak{C}[x] = \mathfrak{L}_2[\mathfrak{L}_1[\hat{d}_1 + \hat{d}_2 + x] + (\partial_k \hat{F}_2, -\partial_k \hat{F}_1)]$ is continuous. Since from [12] we have that $\|(\hat{\omega}, \hat{u}, \hat{v}); \mathcal{V}_\alpha\| \leq \text{const.} \left\|(\hat{F}_2, -\hat{F}_1); \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}} \times \mathcal{B}_{\alpha,\frac{7}{2},\frac{5}{2}}\right\|$, we find with (62) and (63) that the image of $\mathfrak{L}_1$ is arbitrarily small. We then have by linearity of $\mathfrak{L}_2$, that

$\mathfrak{C}$ has a fixed point since $\left\|(\partial_k \hat{F}_2, -\partial_k \hat{F}_1); \mathcal{B}_{\alpha,\frac{3}{2},1} \times \mathcal{B}_{\alpha,\frac{3}{2},2}\right\| < \infty$. This completes the proof of Theorem 9. ■

Theorem 1 now follows by inverse Fourier transform and the decay properties are a direct consequence of the spaces of which $\hat{u}$, $\hat{v}$, $\hat{\omega}$ and $\partial_k \hat{\omega}$ are elements. Indeed, for a function $\hat{f} \in \mathcal{B}^n_{\alpha,p,q}$ with $\alpha > 3$, $n = 0, 1$ and p, $q \geq 0$, we have from the definition of the Fourier transform that

$$\sup_{x \in \mathbb{R}} |f(x,y)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left|\hat{f}(k,y)\right| dk \ , \tag{64}$$

and from the definition of the function spaces that

$$\begin{aligned} \int_{\mathbb{R}} \left|\hat{f}(k,t)\right| \ dk &\leq \left\|\hat{f}_n; \mathcal{B}^n_{\alpha,p,q}\right\| \int_{\mathbb{R}} \frac{1}{\kappa^n} \left(\frac{1}{t^p}\bar{\mu}_\alpha(k,t) + \frac{1}{t^q}\tilde{\mu}_\alpha(k,t)\right) dk \\ &\leq \text{const.} \left\|\hat{f}_n; \mathcal{B}^n_{\alpha,p,q}\right\| \left(\frac{1}{t^{p+(1-n)}} + \frac{1}{t^{q+2(1-n)}}\right) \\ &\leq \frac{\text{const.}}{t^{\min\{p+(1-n),q+2(1-n))}} \left\|\hat{f}_n; \mathcal{B}^n_{\alpha,p,q}\right\| \ . \end{aligned} \tag{65}$$

Combining (64) and (65) we have

$$\sup_{x \in \mathbb{R}} |f(x,y)| \leq \frac{\text{const.}}{y^{\min\{p+(1-n),q+2(1-n))}} \left\|\hat{f}_n; \mathcal{B}^n_{\alpha,p,q}\right\| \ .$$

Finally, we have, using that $(\hat{\omega}, \hat{u}, \hat{v}, \hat{\mathbf{d}}) \in \mathcal{V}_\alpha \times \mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, and that

$$|x\omega(x,y)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\partial_k \omega(k,y)| dk \ ,$$

that

$$\begin{aligned} |y^{3/2} u(x,y)| \leq C_1 \ , \ |y^{3/2} v(x,y)| \leq C_2 \ , \\ |y^3 \omega(x,y)| \leq C_3 \ , \ |yx\omega(x,y)| \leq C_4 \ , \end{aligned}$$

with $C_i \in \mathbb{R}$, for $i = 1, \ldots, 4$, which proves the bound in Theorem 1.

6 Convolution with singularities

We first recall the convolution result from [12].

Proposition 10 (convolution) *Let α, $\beta > 1$, and r, $s \geq 0$ and let a, b be continuous functions from $\mathbb{R}_0 \times [1,\infty)$ to $\mathbb{C}$ satisfying the bounds,*

$$\begin{aligned} |a(k,t)| \leq \mu_{\alpha,r}(k,t) \ , \\ |b(k,t)| \leq \mu_{\beta,s}(k,t) \ . \end{aligned}$$

Then, the convolution $a * b$ is a continuous function from $\mathbb{R} \times [1,\infty)$ to $\mathbb{C}$ and we have the bound

$$|(a * b)(k,t)| \leq \text{const.} \left(\frac{1}{t^r}\mu_{\beta,s}(k,t) + \frac{1}{t^s}\mu_{\alpha,r}(k,t)\right) \ ,$$

uniformly in $t \geq 1$, $k \in \mathbb{R}$.

Since $\partial_k \hat{\omega}$ diverges like $|\kappa|^{-1}$ at $k = 0$ we need to strengthen this result.

Proposition 11 (convolution with $|\kappa|^{-1}$ singularity) *Let $\alpha, \tilde{\beta} > 1$ and $r, \tilde{s} \geq 0$, let a be as in Proposition 10 and $\tilde{b}$ a continuous function from $\mathbb{R}_0 \times [1, \infty)$ to $\mathbb{C}$, satisfying the bound*

$$\left|\tilde{b}(k,t)\right| \leq |\kappa(k)|^{-1} \mu_{\tilde{\beta},\tilde{s}}(k,t) \ ,$$

then the convolution $a * \tilde{b}$ is a continuous function from $\mathbb{R} \times [1,\infty) \to \mathbb{C}$ and we have the bounds

$$\left|(a * \tilde{b})(k,t)\right| \leq \text{const.} \left(\max\left\{ \frac{1}{t^{\frac{\tilde{s}}{2}}}, \frac{1}{t^{\frac{\tilde{s}+r-\tilde{s}'}{2}}} \right\} \mu_{\tilde{\beta},s'}(k,t) + \frac{1}{t^{\frac{\tilde{s}}{2}}} \mu_{\alpha,r}(k,t) \right) \ , \tag{66}$$

$$\left|(a * \tilde{b})(k,t)\right| \leq \text{const.} \left(\max\left\{ \frac{1}{t^{\frac{\tilde{s}}{2}}}, \frac{1}{t^{r-c\tilde{s}'}} \right\} \mu_{\tilde{\beta}+c,\tilde{s}'}(k,t) + \frac{1}{t^{\frac{\tilde{s}}{2}}} \mu_{\alpha,r}(k,t) \right) \ , \tag{67}$$

for $\tilde{s}' \leq \tilde{s}$, and $c \in \left\{\frac{1}{2}, 1\right\}$.

Proof. We drop the $\tilde{}$ to unburden the notation. Continuity is elementary. Since the functions $\mu_{\alpha,r}$ are even in k, we only consider $k \geq 0$. The proof is in two parts, one for $0 \leq k \leq t^{-s'}$ and the other for $t^{-s'} < k$. The first part is valid for both (66) and (67). For $0 \leq k \leq t^{-s'}$, and $\alpha' \geq 0$, we have

$$\begin{aligned}
|(a*b)(k,t)| &\leq \int_{\mathbb{R}} \mu_{\alpha,r}(k',t)|\kappa(k-k')|^{-1}\mu_{\beta,s}(k-k',t)dk' \\
&\leq \sup_{k'\in\mathbb{R}} \left(\mu_{\alpha,r}(k',t)\right) \int_{\mathbb{R}} \frac{t^{\frac{s}{2}}}{|\tilde{k}|^{\frac{1}{2}}} \mu_{\beta,s}(\tilde{k},1) \frac{d\tilde{k}}{t^s} \\
&\leq \frac{\text{const.}}{t^{\frac{s}{2}}} \leq \frac{\text{const.}}{t^{\frac{s}{2}}} \mu_{\alpha',s'}(k,t) \ ,
\end{aligned}$$

where we have used the change of variables $k - k' = \tilde{k}/t^s$. For $k > t^{-s'}$ and $s' \leq s$ we have

$$\begin{aligned}
|(a*b)(k,t)| &\leq \int_{\mathbb{R}} \mu_{\alpha,r}(k',t) \frac{\mu_{\beta,s}(k-k',t)}{|\kappa(k-k')|} dk' \\
&\leq \underbrace{\int_{-\infty}^{k/2} \mu_{\alpha,r}(k',t) \frac{\mu_{\beta,s}(k-k',t)}{|\kappa(k-k')|} dk'}_{:=I_1} + \underbrace{\int_{k/2}^{\infty} \mu_{\alpha,r}(k',t) \frac{\mu_{\beta,s}(k-k',t)}{|\kappa(k-k')|} dk'}_{:=I_2} \ .
\end{aligned}$$

The integral I_2 is the same for (66) and (67),

$$\begin{aligned}
I_2 &= \int_{k/2}^{\infty} \mu_{\alpha,r}(k',t) \frac{1}{|\kappa(k-k')|} \mu_{\beta,s}(k-k',t)dk' \\
&\leq \text{const.}\ \mu_{\alpha,r}(k/2,t) \int_{\mathbb{R}} \frac{t^{\frac{s}{2}}}{|\tilde{k}|^{\frac{1}{2}}} \mu_{\beta,s}(\tilde{k},1) \frac{d\tilde{k}}{t^s} \\
&\leq \text{const.} \frac{1}{t^{s/2}} \mu_{\alpha,r}(k,t) \ ,
\end{aligned}$$

where again we have used the change of variables $k - k' = \tilde{k}/t^s$. To compute the integral I_1 we use that

$$\begin{aligned}
\mu_{\alpha,s}(k,t) &\leq \mu_{\alpha,s'}(k,t) \ , \\
\mu_{\alpha,s}(k,t) \cdot \mu_{\beta,s}(k,t) &\leq \text{const.} \mu_{\alpha+\beta,s}(k,t) \ ,
\end{aligned}$$

and, for $k > t^{-s'}$,

$$\begin{aligned}
\frac{1}{t^{\frac{s'}{2}}} \frac{1}{|\kappa(k)|} &\leq \frac{\text{const.}}{t^{\frac{s'}{2}}|k|^{1/2}} \leq \frac{\text{const.}}{2t^{\frac{s'}{2}}|k|^{1/2}} \leq \frac{\text{const.}}{1 + (t^{s'}|k|)^{\frac{1}{2}}} \leq \mu_{\frac{1}{2},s'}(k,t) \ , \\
\frac{1}{t^{s'}} \frac{1}{|\kappa(k)|} &\leq \frac{\text{const.}}{t^{s'}\left(|k|^{1/2} + |k|\right)} \leq \frac{\text{const.}}{t^{s'/2} + |k|t^{s'}} \leq \frac{\text{const.}}{1 + |k|t^{s'}} \leq \mu_{1,s'}(k,t) \ .
\end{aligned}$$

To prove (66), we note that

$$
\begin{aligned}
I_1^{(66)} &\leq \int_{-\infty}^{k/2} \frac{\mu_{\alpha,r}(k',t)}{|k'|^{\frac{1}{2}}} \frac{|k'|^{\frac{1}{2}}}{|k-k'|^{\frac{1}{2}}} \mu_{\beta,s}(k-k',t)dk' \\
&\leq \int_{-\infty}^{k/2} \frac{\mu_{\alpha,r}(k',t)}{|k'|^{\frac{1}{2}}} \frac{|k|^{\frac{1}{2}}}{|k-k'|^{\frac{1}{2}}} \mu_{\beta,s}(k-k',t)dk' \\
&+ \int_{-\infty}^{k/2} \frac{\mu_{\alpha,r}(k',t)}{|k'|^{\frac{1}{2}}} \frac{|k-k'|^{\frac{1}{2}}}{|k-k'|^{\frac{1}{2}}} \mu_{\beta,s}(k-k',t)dk' \\
&\leq \frac{|k|^{\frac{1}{2}}}{|k/2|^{\frac{1}{2}}} \mu_{\beta,s}(k/2,t) \int_{-\infty}^{k/2} \frac{\mu_{\alpha,r}(k',t)}{|k'|^{\frac{1}{2}}} dk' \\
&+ \int_{-\infty}^{k/2} \frac{\mu_{\alpha,r}(k',t)}{|k'|^{\frac{1}{2}}} \frac{1}{|k-k'|^{\frac{1}{2}}} \frac{\text{const.}}{t^{s/2}} \mu_{\beta-\frac{1}{2},s}(k-k',t)dk' \\
&\leq \frac{\text{const.}}{|k|^{\frac{1}{2}}} \frac{1}{t^{s/2}} \mu_{\beta-\frac{1}{2},s}(k,t) \int_{-\infty}^{k/2} \frac{\mu_{\alpha,r}(k',t)}{|k'|^{\frac{1}{2}}} dk' \\
&\leq \text{const.} \frac{t^{\frac{s'}{2}}}{t^{\frac{s}{2}}} \frac{1}{t^{\frac{s'}{2}} |k|^{\frac{1}{2}}} \mu_{\beta-\frac{1}{2},s}(k,t) \frac{1}{t^{\frac{r}{2}}} \\
&\leq \text{const.} \frac{t^{\frac{s'}{2}}}{t^{\frac{s}{2}}} \mu_{\frac{1}{2},s'}(k,t)\, \mu_{\beta-\frac{1}{2},s}(k,t) \frac{1}{t^{\frac{r}{2}}} \leq \frac{\text{const.}}{t^{\frac{s+r-s'}{2}}} \mu_{\beta,s'}(k,t) \ ,
\end{aligned}
$$

where we have used the family of inequalities

$$
|k|^{\rho} \mu_{\alpha,r}(k,t) \leq \text{const.} \frac{1}{t^{\rho r}} \mu_{\alpha-p,r}(k,t) \ \ ,\forall \rho > 0 \ . \tag{68}
$$

Finally, to prove (67), we note that

$$
\begin{aligned}
I_1^{(67)} &\leq \int_{-\infty}^{k/2} \mu_{\alpha,r}(k',t) \frac{1}{|\kappa(k-k')|} \mu_{\beta,s}(k-k',t)dk' \\
&\leq \frac{t^{cs'}}{t^{cs'}} \frac{1}{|\kappa(k/2)|} \mu_{\beta,s}(k/2,t) \int_{\mathbb{R}} \mu_{\alpha,r}(k',t)dk' \\
&\leq \text{const.} t^{cs'} \mu_{c,s'}(k, t\mu_{\beta,s}(k,t) \int_{\mathbb{R}} \mu_{\alpha,r}(k',t)dk' \\
&\leq \text{const.} \frac{1}{t^{r-cs'}} \mu_{\beta+c,s'}(k,t) \ .
\end{aligned}
$$

Collecting the bounds on the integrals $I_1^{(66)}$, $I_1^{(67)}$ and I_2 proves the claim in Proposition 11. ■

Corollary 12 *Let $\alpha > 2$ and, for $i = 1, 2$, $p_i, q_i \geq 0$. Let $f \in \mathcal{B}_{\alpha,p_1,q_1}$ and $g \in \mathcal{D}^1_{\alpha-1,p_2,q_2}$. Let*

$$
\begin{aligned}
p &= \min\{p_1 + p_2 + \frac{1}{2}, p_1 + q_2 + 1, q_1 + p_2 + \frac{1}{2}\} \ , \\
q &= \min\{q_1 + q_2 + 1, q_1 + p_2 + \frac{1}{2}\} \ .
\end{aligned}
$$

Then $f * g \in \mathcal{B}_{\alpha,p,q}$, and there exists a constant C, depending only on α, such that

$$
\|f * g; \mathcal{B}_{\alpha,p,q}\| \leq C \ \|f; \mathcal{B}_{\alpha,p_1,q_1}\| \cdot \|g; \mathcal{D}^1_{\alpha-1,p_2,q_2}\| \ .
$$

Proof. We consider the three cases $c \in \{0, \frac{1}{2}, 1\}$. Let $\tilde{g}$ be a function in $\mathcal{B}^1_{\tilde{\alpha},\tilde{p},\tilde{q}}$, with $\tilde{\alpha} = \alpha - c$, $\tilde{p}, \tilde{q} \geq 0$. The convolution product $f * \tilde{g}$ is in each case bounded by a function in $\mathcal{B}^1_{\alpha,p,q}$ with p and q given by:

- if $c = 0$, $p = \min\{p_1 + \tilde{p} + \frac{1}{2}, p_1 + \tilde{q} + 1, q_1 + \tilde{p} + \frac{1}{2}\}$, $q = \min\{q_1 + \tilde{q} + 1, q_1 + \tilde{p} + \frac{1}{2}\}$,

- if $c = \frac{1}{2}$, $p = \min\{p_1 + \tilde{p} + \frac{1}{2}, p_1 + \tilde{q} + \frac{1}{2}, q_1 + \tilde{p} + \frac{1}{2}\}$, $q = \min\{q_1 + \tilde{q} + 1, q_1 + \tilde{p} + \frac{1}{2}\}$,
- if $c = 1$, $p = \min\{p_1 + \tilde{p} + 0, p_1 + \tilde{q} + 0, q_1 + \tilde{p} + \frac{1}{2}\}$, $q = \min\{q_1 + \tilde{q} + 0, q_1 + \tilde{p} + \frac{1}{2}\}$.

These are consequences of Proposition 11. Using equation (66) for the first case and equation (67) for the following two cases, and choosing $s' = 1$ to bound the term $\frac{1}{t^{p_1}}\bar{\mu}_\alpha * \frac{1}{t^{\tilde{q}}}\tilde{\mu}_{\tilde{\alpha}}$. It is now clear that for a function in $\mathcal{D}^1_{\alpha-1,p_2,q_2}$, the terms that yield the lowest p and q are covered by the $c = 0$ case above, because what is lost in the bounds on convolution due to lower $\tilde{\alpha}$ is gained through higher values of $\tilde{p}$ and $\tilde{q}$ by definition of the space $\mathcal{D}^1_{\alpha-1,p_2,q_2}$. This corollary allows to streamline notations and shorten calculations throughout the paper. ■

7 Bounds on $\hat{\mathbf{d}}$

We present some elementary inequalities and expressions used throughout this section. Throughout the calculations we will use without further mention, that for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \leq 0$ and $N \in \mathbb{N}_0$,

$$\left| \frac{e^z - \sum_{n=0}^{N} \frac{1}{n!} z^n}{z^{N+1}} \right| \leq \text{const.} \ ,$$

and for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$

$$\left| \frac{e^z - \sum_{n=0}^{N} \frac{1}{n!} z^n}{z^{N+1}} \right| \leq \text{const.} e^{\mathrm{Re}(z)} \ .$$

We also have that

$$\partial_k \kappa = \frac{2k - i}{2\kappa} \ .$$

By definition of the norm on $\mathcal{D}^1_{\alpha,p,q}$ we must bound $\kappa \partial_k \hat{\omega}$. We thus bound all the terms $\kappa \partial_k \hat{\omega}_{l,n,m}$, with $l = 1, 2, 3$, $n = 1, 2, 3$ and $m = 0, 1$ (see definitions (38), (39)-(41) and (32)-(49)). This requires a good deal of book-keeping to track what happens to α, p, and q. Some of it may be spared when one realizes that all losses in α occur when applying (68) where there are explicit factors $|k|^c$ with $c = \{\frac{1}{2}, 1\}$, which automatically brings forth a structure satisfying the conditions of Corollary 12. This allows us to show that each component $\partial_k \hat{\omega}_{l,n,m}$ is an element of a $\mathcal{D}^1_{\alpha-1,p,q}$.

From (61) we obtain, for $i = 0, 1$,

$$\left| \hat{Q}_i (k, s) \right| \leq \left\| \hat{Q}_i ; \mathcal{B}_{\alpha, \frac{7}{2}, \frac{5}{2}} \right\| \left(\frac{1}{s^{\frac{7}{2}}} \bar{\mu}_\alpha + \frac{1}{s^{\frac{5}{2}}} \tilde{\mu}_\alpha \right) \ ,$$

which we will use throughout without further mention. We also make use of equations (56) and (68) without explicit mention throughout these proofs.

The bounds for the terms $n = 2$ take advantage of the fact that, for $1 \leq t < 2$,

$$\bar{\mu}_\alpha(k, t) \leq \text{const.} \ \tilde{\mu}_\alpha(k, t) \leq \text{const.}$$

and, for $t \geq 2$ and $\alpha' > 0$,

$$e^{\Lambda_-(t-1)} \mu_{\alpha,r}(k, t) \leq \text{const.} \ e^{\Lambda_-(t-1)} \leq \text{const.} \ \tilde{\mu}_{\alpha'}(k, t) \ ,$$

so that the inequality

$$e^{\Lambda_-(t-1)} \mu_{\alpha,r}(k, t) \leq \text{const.} \ \tilde{\mu}_\alpha(k, t) \tag{69}$$

holds for all t and $\alpha > 0$.

7.1 Bounds on $\hat{d}_1$

To show that $\hat{d}_1 = \sum_{m=0}^{1}\sum_{n=1}^{3} \partial_k \hat{\omega}_{1,n,m}$ is in $\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, which constitutes the first part of Proposition 6, we first need to recall a proposition proved in [12].

Proposition 13 *Let $f_{n,m}$ be as given in Section 3. Then we have the bounds*

$$|f_{1,0}(k,\sigma)| \leq \text{const. } e^{|\Lambda_-|\sigma} \min\{|\Lambda_-|, |\Lambda_-|^3\sigma^2\} \ , \tag{70}$$

$$|f_{2,0}(k,\sigma)| \leq \text{const. } (|k| + |k|^{1/2}) e^{-|k|\sigma} \ , \tag{71}$$

$$|f_{3,0}(k,\sigma)| \leq \text{const. } e^{\Lambda_-\sigma} \min\{1, |\Lambda_-|^2\} \ , \tag{72}$$

$$|f_{1,1}(k,\sigma)| \leq \text{const. } (1 + |\Lambda_-|) e^{|\Lambda_-|\sigma} \min\{1, |\Lambda_-|\sigma\} \ , \tag{73}$$

$$|f_{2,1}(k,\sigma)| \leq \text{const. } (1 + |k|)\, e^{-|k|\sigma} \ , \tag{74}$$

$$|f_{3,1}(k,\sigma)| \leq \text{const. } e^{\Lambda_-\sigma} \min\{1, |\Lambda_-|\} \ , \tag{75}$$

uniformly in $\sigma \geq 0$ and $k \in \mathbb{R}_0$.

We then note that

$$\left|\kappa \partial_k \check{K}_n(k,\tau)\right| = \left|\frac{1}{2}\tau\kappa\frac{2k-i}{2\kappa}e^{-\kappa\tau}\right| \leq \text{const. } \tau(1+|k|)e^{\Lambda_-\tau} \ , \text{ for } n = 1,2 \ ,$$

$$\left|\kappa \partial_k \check{K}_3(k,\tau)\right| = \left|\frac{1}{2}\tau\kappa\frac{2k-i}{2\kappa}(e^{\kappa\tau} + e^{-\kappa\tau})\right| \leq \text{const. } \tau(1+|k|)(e^{|\Lambda_-|\tau} + e^{\Lambda_-\tau}) \ .$$

The bound on the function $\kappa\partial_k\hat{\omega}_{1,1,0}$ uses (70) and Propositions 15 and 16, leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{1,1,0}| &= \left|\kappa\frac{1}{2}\partial_k e^{-\kappa\tau}\int_1^t \check{f}_{1,0}\,(k,\sigma)\,\hat{Q}_0\,(k,s)\,ds\right| \\
&\leq \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}\int_1^t e^{|\Lambda_-|\sigma}\min\{|\Lambda_-|, |\Lambda_-|^3\sigma^2\}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha + \frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right) ds \\
&\leq \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}\int_1^{\frac{t+1}{2}} e^{|\Lambda_-|\sigma}|\Lambda_-|^3\sigma^2\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha + \frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right) ds \\
&+ \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}\int_{\frac{t+1}{2}}^t e^{|\Lambda_-|\sigma}|\Lambda_-|\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha + \frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right) ds \\
&\leq \text{const. } (1+|k|)\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha + \frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha\right) \ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{1,1,0} \in \mathcal{D}^1_{\alpha-1,\frac{5}{2},\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{1,2,0}$ uses (71), Proposition 18 and (69), leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{1,2,0}| &= \left|\kappa\frac{1}{2}\partial_k e^{-\kappa\tau}\int_t^\infty \check{f}_{2,0}(k,s-1)\hat{Q}_0(k,s)ds\right| \\
&\leq \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}e^{|k|\tau}\int_t^\infty (|k|^{\frac{1}{2}} + |k|)e^{-|k|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha + \frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right) ds \\
&\leq \text{const. } (1+|k|)e^{\Lambda_-\tau}\left(\frac{1}{t^2}\bar{\mu}_\alpha + \frac{1}{t^1}\tilde{\mu}_\alpha\right) \leq \text{const. } (1+|k|)\frac{1}{t^1}\tilde{\mu}_\alpha \ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{1,2,0} \in \mathcal{D}^1_{\alpha-1,\infty,1}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{1,3,0}$ uses (72) and Proposition 17, leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{1,3,0}| &= \left|\frac{1}{2}\kappa\partial_k(e^{\kappa\tau}-e^{-\kappa\tau})\int_t^{\infty}\check{f}_{3,0}(k,s-1)\hat{Q}_0(k,s)ds\right| \\
&\leq \text{const. } \tau(1+|k|)(e^{|\Lambda_-|\tau}+e^{\Lambda_-\tau})\int_t^{\infty}\min\{1,|\Lambda_-|\}e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } \tau e^{|\Lambda_-|\tau}\int_t^{\infty}(1+|\Lambda_-|)\min\{1,|\Lambda_-|\}e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. }\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha\right) ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{1,3,0}\in\mathcal{D}^1_{\alpha-1,\frac{5}{2},\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{1,1,1}$ uses (73) and Propositions 15 and 16, leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{1,1,1}| &= \left|\kappa\frac{1}{2}\partial_k e^{-\kappa\tau}\int_1^{t}\check{f}_{1,1}(k,s-1)\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}\int_1^{t}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}\min\{1,|\Lambda_-|\sigma\}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}\int_1^{t}e^{|\Lambda_-|\sigma}|\Lambda_-|\sigma\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&+ \text{const. } \tau(1+|k|)e^{\Lambda_-\tau}\int_1^{t}|\Lambda_-|e^{|\Lambda_-|\sigma}\min\{1,|\Lambda_-|\sigma\}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|k|)\left(\tilde{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha\right) ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{1,1,1}\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{1,2,1}$ uses (74), Proposition 18 and (69), leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{1,2,1}| &= \left|\frac{1}{2}\kappa\partial_k e^{-\kappa\tau}\int_t^{\infty}\check{f}_{2,1}(k,s-1)\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } (1+|k|)\tau e^{\Lambda_-\tau}\int_t^{\infty}(1+|k|)e^{-|k|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|k|)e^{\Lambda_-\tau}\left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha\right)\leq \text{const. } (1+|k|)\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{1,2,1}\in\mathcal{D}^1_{\alpha-1,\infty,\frac{1}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{1,3,1}$ uses (75) and Proposition 17, leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{1,3,1}| &= \left|\kappa\frac{1}{2}\partial_k(e^{\kappa\tau}-e^{-\kappa\tau})\int_t^{\infty}\check{f}_{3,1}\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } \tau(1+|k|)(e^{|\Lambda_-|\tau}+e^{\Lambda_-\tau})\int_t^{\infty}e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } \tau(e^{|\Lambda_-|\tau}+e^{\Lambda_-\tau})\int_t^{\infty}(1+|\Lambda_-|)e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. }\left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha(k,t)+\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha(k,t)\right) ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{1,3,0}\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},\frac{1}{2}}$.

Collecting the bounds we find that $\hat{d}_1\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, which completes the first part of the proof of Proposition 6.

7.2 Bounds on $\hat{d}_2$

To show that $\hat{d}_2 = \sum_{m=0}^{1} \sum_{n=1}^{3} \partial_k \hat{\omega}_{2,n,m}$ is in $\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, which constitutes the second part of Proposition 6, we first need to show bounds on the functions $\partial_k \check{f}_{n,m}$.

Proposition 14 *Let $\partial_k \check{f}_{n,m}$ be as given in Section 3. Then we have the bounds*

$$\left|\kappa\partial_k \check{f}_{1,0}(k,\sigma)\right| \leq \text{const. } \min\{(1+|\Lambda_-|\sigma),(s+|\Lambda_-|)|\Lambda_-|^2\sigma\} e^{|\Lambda_-|\sigma} \ , \tag{76}$$
$$\left|\kappa\partial_k \check{f}_{2,0}(k,\sigma)\right| \leq \text{const. } (|k|^{\frac{1}{2}} + |k|^2)\sigma e^{-|k|\sigma} \ , \tag{77}$$
$$\left|\kappa\partial_k \check{f}_{3,0}(k,\sigma)\right| \leq \text{const. } (1+|\Lambda_-|\sigma) e^{\Lambda_-\sigma} \ , \tag{78}$$
$$\left|\kappa\partial_k \check{f}_{1,1}(k,\sigma)\right| \leq \text{const. } (1+|\Lambda_-|^2)\sigma e^{|\Lambda_-|\sigma} \ , \tag{79}$$
$$\left|\kappa\partial_k \check{f}_{2,1}(k,\sigma)\right| \leq \text{const. } (1+|k|^2)\sigma e^{-|k|\sigma} \ , \tag{80}$$
$$\left|\kappa\partial_k \check{f}_{3,1}(k,\sigma)\right| \leq \text{const. } (1+|\Lambda_-|)\sigma e^{\Lambda_-\sigma} \ , \tag{81}$$

uniformly in $\sigma \geq 0$ and $k \in \mathbb{R}_0$.

Proof. We multiply (44)-(49) by κ and bound the products. The function $\kappa\partial_k \check{f}_{1,0}$ is bounded in two ways. We have a straightforward bound

$$\left|\kappa\partial_k \check{f}_{1,0}(k,\sigma)\right| \leq \text{const. } (1+|\Lambda_-|\sigma) e^{|\Lambda_-|\sigma} \ .$$

Since leading terms cancel, we get

$$\begin{aligned}
\left|\kappa\partial_k \check{f}_{1,0}(k,\sigma)\right| &\leq \left|\frac{i}{2}\left(e^{\kappa\sigma} + e^{-\kappa\sigma} - 2e^{-|k|\sigma}\right) - \frac{ik^2}{2\kappa^2}\left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right) + 2\frac{k^2+|k|\kappa}{k}(e^{-|k|\sigma} - e^{-\kappa\sigma})\right| \\
&+ \left| i\frac{k^2+\kappa^2}{2\kappa}\left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right)\sigma + \frac{k^2+|k|\kappa}{k}\frac{k^2+\kappa^2}{\kappa} e^{-\kappa\sigma}\sigma - 2\kappa\frac{k^2+|k|\kappa}{k} e^{-|k|\sigma}\sigma\right| \\
&\leq \text{const. } |(e^{\kappa\sigma} - 1 - \kappa\sigma) + (e^{-\kappa\sigma} - 1 + \kappa\sigma) - 2(e^{-|k|\sigma} - 1)| \\
&+ \text{const. } \left|\frac{k^2}{\kappa^2}\left((e^{\kappa\sigma} - 1) - (e^{-\kappa\sigma} - 1)\right)\right| + \text{const. } |\Lambda_-||(e^{-|k|\sigma} - 1) - (e^{-\kappa\sigma} - 1)| \\
&+ \text{const. } \left|\frac{k^2+\kappa^2}{2\kappa}\left((e^{\kappa\sigma} - 1) - (e^{-\kappa\sigma} - 1)\right)\sigma\right| \\
&+ \text{const. } \left|\frac{k^2+|k|\kappa}{k}\frac{k^2+\kappa^2}{\kappa} e^{-\kappa\sigma}\sigma\right| + \text{const. } \left|\kappa\frac{k^2+|k|\kappa}{k} e^{-|k|\sigma}\sigma\right| \\
&\leq \text{const. } (|\Lambda_-|^2\sigma^2 + |k|\sigma) e^{|\Lambda_-|\sigma} + \text{const. } |\Lambda_-|^3\sigma e^{|\Lambda_-|\sigma} + \text{const. } |\Lambda_-|^2\sigma e^{|\Lambda_-|\sigma} \\
&+ \text{const. } |\Lambda_-|^2\sigma^2 e^{|\Lambda_-|\sigma} + \text{const. } |\Lambda_-|^2\sigma e^{|\Lambda_-|\sigma} + \text{const. } |\Lambda_-|^2\sigma e^{|\Lambda_-|\sigma} \\
&\leq \text{const. } (s+|\Lambda_-|)|\Lambda_-|^2\sigma e^{|\Lambda_-|\sigma} \ .
\end{aligned}$$

Then we have

$$\left|\kappa\partial_k \check{f}_{1,0}(k,\sigma)\right| \leq \text{const. } \min\{(1+|\Lambda_-|\sigma),(s+|\Lambda_-|)|\Lambda_-|^2\sigma\} e^{|\Lambda_-|\sigma} \ ,$$

which proves (76).

To bound $\kappa\partial_k \check{f}_{2,0}(k,\sigma)$ we use that, since $|k| \leq \mathrm{Re}(\kappa)$ for all k,

$$\begin{aligned}
\left|e^{-|k|\sigma} - e^{-\kappa\sigma}\right| &\leq \text{const. } e^{-|k|\sigma}\left|1 - e^{(|k|-\kappa)\sigma}\right| \\
&\leq \text{const. } e^{-|k|\sigma}\,||k| - \kappa|\,\sigma \\
&\leq \text{const. } (|k|^{\frac{1}{2}} + |k|)\sigma e^{-|k|\sigma} \ ,
\end{aligned} \tag{82}$$

such that

$$
\begin{aligned}
\left|\kappa\partial_k \check{f}_{2,0}(k,\sigma)\right| &\leq \left|\frac{(|k|+\kappa)^2}{k}\left(e^{-|k|\sigma}-e^{-\kappa\sigma}\right)-2\frac{\kappa+|k|}{k}\left(|k|\kappa e^{-|k|\sigma}-\frac{k^2+\kappa^2}{2}e^{-\kappa\sigma}\right)\sigma\right| \\
&\leq \text{const. } (1+|k|)(|k|^{\frac{1}{2}}+|k|)e^{-|k|\sigma}\sigma+\text{const. } (|k|+|k|^2)e^{-|k|\sigma}\sigma \\
&+\text{const. } (|k|^{\frac{1}{2}}+|k|^2)e^{-|k|\sigma}\sigma \\
&\leq \text{const. } (|k|^{\frac{1}{2}}+|k|^2)\sigma e^{-|k|\sigma}\ ,
\end{aligned}
$$

which gives (77).

To bound $\kappa\partial_k \check{f}_{3,0}\,(k,\sigma)$ we have the straightforward bound

$$
\begin{aligned}
\left|\kappa\partial_k \check{f}_{3,0}\,(k,\sigma)\right| &\leq \left|\kappa\frac{k}{2\kappa^3}e^{-\kappa\sigma}\right|+\left|\kappa\frac{k^2+\kappa^2}{2\kappa^2}\sigma e^{-\kappa\sigma}\right| \\
&\leq \text{const. } (1+|\Lambda_-|)\sigma e^{\Lambda_-\sigma}\ ,
\end{aligned}
$$

which yields (78).

To bound $\kappa\partial_k \check{f}_{1,1}\,(k,\sigma)$ we have

$$
\begin{aligned}
\left|\kappa\partial_k \check{f}_{1,1}(k,\sigma)\right| &\leq \left|i\frac{(|k|+\kappa)^2}{|k|}(e^{-|k|\sigma}-e^{-\kappa\sigma})\right|+\left|\frac{k^2+\kappa^2}{2k}\left(e^{\kappa\sigma}+e^{-\kappa\sigma}\right)\sigma\right| \\
&+\left|2i\frac{k^2+|k|\kappa}{k^2}\left(\frac{k^2+\kappa^2}{2}e^{-\kappa\sigma}-|k|\kappa e^{-|k|\sigma}\right)\sigma\right| \\
&\leq \text{const. } (1+|k|)(|k|+|\Lambda_-|)\sigma+\text{const. } (1+|k|)\sigma e^{|\Lambda_-|\sigma} \\
&+\text{const. } |\Lambda_-|((1+|k|)+|\Lambda_-|)\sigma \leq \text{const. } (1+|\Lambda_-|^2)\sigma e^{|\Lambda_-|\sigma}\ ,
\end{aligned}
$$

and thus we have (79).

To bound $\kappa\partial_k \check{f}_{2,1}\,(k,\sigma)$ we use (82) to bound

$$
\begin{aligned}
\left|\kappa\partial_k \check{f}_{2,1}\,(k,\sigma)\right| &\leq \left|i\frac{(|k|+\kappa)^2}{|k|}(e^{-\kappa\sigma}-e^{-|k|\sigma})\right|+\left|i(|k|+\kappa)\kappa\frac{k^2+\kappa^2}{k^2}e^{-\kappa\sigma}\sigma\right| \\
&+\left|2i\kappa\left(|k|+\kappa\right)e^{-|k|\sigma}\sigma\right| \\
&\leq \text{const. } (1+|k|)(|k|^{\frac{1}{2}}+|k|)\sigma e^{-|k|\sigma}+\text{const. } (|k|+|k|^2)(1+|k|^{-1})e^{-|k|\sigma}\sigma \\
&+\text{const. } (|k|+|k|^2)e^{-|k|\sigma}\sigma \leq \text{const. } (1+|k|^2)\sigma e^{-|k|\sigma}
\end{aligned}
$$

which leads to (80).

Finally, To bound $\kappa\partial_k \check{f}_{3,1}\,(k,\sigma)$ we have the straightforward bound

$$
\left|\kappa\partial_k \check{f}_{3,1}\,(k,\sigma)\right| \leq \left|\frac{k^2+\kappa^2}{2k}e^{-\kappa\sigma}\sigma\right| \leq \text{const. } (1+|\Lambda_-|)\sigma e^{\Lambda_-\sigma}\ ,
$$

and therefore we have (81). This completes the proof of Proposition 14. ■

We may now bound $\hat{d}_2$. The bound on the function $\kappa\partial_k\hat{\omega}_{2,1,0}$ uses (76) and Propositions 15 and 16, leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{2,1,0}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_1^t \kappa\partial_k\check{f}_{1,0}\left(k,\sigma\right)\hat{Q}_0\left(k,s\right)ds\right| \\
&\leq \text{const.}\ e^{\Lambda_-\tau}\int_1^t \min\{(1+|\Lambda_-|\sigma),(s+|\Lambda_-|)|\Lambda_-|^2\sigma\}e^{|\Lambda_-|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\ e^{\Lambda_-\tau}\int_1^{\frac{t+1}{2}} (s+|\Lambda_-|)|\Lambda_-|^2\sigma e^{|\Lambda_-|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&+ \text{const.}\ e^{\Lambda_-\tau}\int_{\frac{t+1}{2}}^t (1+|\Lambda_-|\sigma)e^{|\Lambda_-|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\ (1+|\Lambda_-|)\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha+\text{const.}\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha\right)\ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{2,1,0}\in\mathcal{D}^1_{\alpha-1,\frac{5}{2},\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{2,2,0}$ uses (77), Proposition 18 and (69), leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{2,2,0}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_t^\infty \kappa\partial_k\check{f}_{2,0}(k,s-1)\hat{Q}_0(k,s)ds\right| \\
&\leq \text{const.}\ e^{\Lambda_-\tau}e^{|k|\tau}\int_t^\infty (|k|^{\frac{1}{2}}+|k|^2)\sigma e^{-|k|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\ (1+|k|)e^{\Lambda_-\tau}\left(\frac{1}{t^2}\bar{\mu}_\alpha+\frac{1}{t^1}\tilde{\mu}_\alpha\right)\leq \text{const.}\ (1+|k|)\frac{1}{t^1}\tilde{\mu}_\alpha\ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{2,2,0}\in\mathcal{D}^1_{\alpha-1,\infty,1}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{2,3,0}$ uses (78) and Proposition 17, leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{2,3,0}| &= \left|\frac{1}{2}(e^{\kappa\tau}-e^{-\kappa\tau})\int_t^\infty \kappa\partial_k\check{f}_{3,0}(k,s-1)\hat{Q}_0(k,s)ds\right| \\
&\leq \text{const.}\ e^{|\Lambda_-|\tau}\int_t^\infty (1+|\Lambda_-|\sigma)e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\ e^{|\Lambda_-|\tau}\int_t^\infty e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&+ \text{const.}\ e^{|\Lambda_-|\tau}\int_t^\infty |\Lambda_-|e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{5}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{3}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha\right)\ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{2,3,0}\in\mathcal{D}^1_{\alpha-1,\frac{5}{2},\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{2,1,1}$ uses (79) and Propositions 15 and 16, leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{2,1,1}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_1^t \kappa\partial_k\check{f}_{1,1}(k,s-1)\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const.}\ e^{\Lambda_-\tau}\int_1^t (1+|\Lambda_-|^2)\sigma e^{|\Lambda_-|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\left(\tilde{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha\right) \\
&+ \text{const.}\ \frac{1}{t^1}\tilde{\mu}_\alpha+\text{const.}\ |\Lambda_-|\left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha\right)\ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{2,1,1}\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{2,2,1}$ uses (80), Proposition 18 and (69), leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{2,2,1}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_t^\infty \kappa\partial_k\check{f}_{2,1}(k,s-1)\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } e^{\Lambda_-\tau}e^{|k|\tau}\int_t^\infty (1+|k|^2)\sigma e^{-|k|\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|k|)e^{\Lambda_-\tau}\left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha\right) \leq \text{const. } (1+|k|)\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha \ ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{2,2,1} \in \mathcal{D}^1_{\alpha-1,\infty,\frac{1}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{2,3,1}$ uses (81) and Proposition 17, leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{2,3,1}| &= \left|\frac{1}{2}(e^{\kappa\tau}-e^{-\kappa\tau})\int_t^\infty \kappa\partial_k\check{f}_{3,1}\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } (e^{|\Lambda_-|\tau}+e^{\Lambda_-\tau})\int_t^\infty (1+|\Lambda_-|)\sigma e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{7}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } \left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{1}{2}}}\tilde{\mu}_\alpha\right) \ ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{2,3,1} \in \mathcal{D}^1_{\alpha-1,\frac{3}{2},\frac{1}{2}}$.

Collecting the bounds we have that $\hat{d}_2 \in \mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, which completes the second part of the proof of Proposition 6.

7.3 Bounds on $\hat{d}_3$

We prove the bounds on $\hat{d}_3$ needed to complete the proof of Lemma 8. For compatibility with the maps $\mathfrak{L}_1$ and $\mathfrak{L}_2$ we will bound $\kappa\hat{d}_3$ instead of $\hat{d}_3$. Throughout this proof we will use without further mention the bounds

$$
\begin{aligned}
\left|\partial_k\hat{Q}_0\,(k,s)\right| &\leq \left\|\partial_k\hat{Q}_0\right\|\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^1}\tilde{\mu}_\alpha\right) \ , \\
\left|\partial_k\hat{Q}_1\,(k,s)\right| &\leq \left\|\partial_k\hat{Q}_1\right\|\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right) \ .
\end{aligned}
$$

The bound on the function $\kappa\partial_k\hat{\omega}_{3,1,0}$ uses (70) and Propositions 15 and 16, leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{3,1,0}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_1^t \check{f}_{1,0}\,(k,\sigma)\,\kappa\partial_k\hat{Q}_0\,(k,s)\,ds\right| \\
&\leq \text{const. } |\Lambda_-|e^{\Lambda_-\tau}\int_1^t e^{|\Lambda_-|\sigma}\min\{|\Lambda_-|,|\Lambda_-|^3\sigma^2\}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } |\Lambda_-|e^{\Lambda_-\tau}\int_1^{\frac{t+1}{2}} e^{|\Lambda_-|\sigma}|\Lambda_-|^3\sigma^2\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s}\tilde{\mu}_\alpha\right)ds \\
&+ \text{const. } |\Lambda_-|e^{\Lambda_-\tau}\int_{\frac{t+1}{2}}^t e^{|\Lambda_-|\sigma}|\Lambda_-|\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } |\Lambda_-|\left(\frac{1}{t^3}\tilde{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^1}\tilde{\mu}_\alpha\right) \ ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{3,1,0} \in \mathcal{D}^1_{\alpha-1,\frac{3}{2},1}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{3,2,0}$ uses (71), (69) and Proposition 18, which, to be applicable,

requires first the use of (68) to trade a $|k|$ for an s^{-1} multiplying $\bar{\mu}_\alpha$ and $\tilde{\mu}_\alpha$. We then have

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{3,2,0}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_t^\infty \check{f}_{2,0}(k,s-1)\kappa\partial_k\hat{Q}_0(k,s)ds\right| \\
&\leq \text{const. } e^{\Lambda_-\tau}\int_t^\infty (|k|+|k|^{\frac{1}{2}})(|k|^{\frac{1}{2}}+|k|)e^{-|k|\sigma}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } e^{\Lambda_-\tau}e^{|k|\tau}\int_t^\infty |k|e^{-|k|\sigma}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^{\frac{5}{2}}}\bar{\mu}_{\alpha-1}\right)ds \\
&+ \text{const. } e^{\Lambda_-\tau}e^{|k|\tau}\int_t^\infty (1+|k|)\,e^{-|k|\sigma}\frac{1}{s^3}\tilde{\mu}_{\alpha-1}ds \\
&\leq \text{const. } e^{\Lambda_-\tau}\left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_{\alpha-1}+\frac{1}{t^2}\tilde{\mu}_{\alpha-1}\right) \\
&\leq \text{const. } \left(\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha+\frac{1}{t^2}\tilde{\mu}_{\alpha-1}\right) \quad ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{3,2,0}\in\mathcal{D}^1_{\alpha-1,\infty,\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{3,3,0}$ uses (72) and Proposition 17, which, to be applicable, requires first the use of (68) to trade a $|\Lambda_-|$ for a $s^{-1/2}$ multiplying $\tilde{\mu}_\alpha$. We then have

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{3,3,0}| &= \left|\frac{1}{2}(e^{\kappa\tau}-e^{-\kappa\tau})\int_t^\infty \check{f}_{3,0}(k,s-1)\kappa\partial_k\hat{Q}_0(k,s)ds\right| \\
&\leq \text{const. } e^{|\Lambda_-|\tau}\int_t^\infty \min\{1,|\Lambda_-|\}e^{\Lambda_-\sigma}|\Lambda_-|\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } e^{|\Lambda_-|\tau}\int_t^\infty |\Lambda_-|e^{\Lambda_-\sigma}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_{\alpha-\frac{1}{2}}+\frac{1}{s^3}\tilde{\mu}_{\alpha-1}\right)ds \\
&\leq \text{const. } \left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^2}\tilde{\mu}_{\alpha-\frac{1}{2}}+\frac{1}{t^3}\tilde{\mu}_{\alpha-1}\right) \quad ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{3,3,0}\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{3,1,1}$ uses (73) and Propositions 15 and 16, leading to

$$
\begin{aligned}
|\kappa\partial_k\hat{\omega}_{3,1,1}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_1^t \check{f}_{1,1}(k,s-1)\kappa\partial_k\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } e^{\Lambda_-\tau}\int_1^t (1+|\Lambda_-|)e^{|\Lambda_-|\sigma}\min\{1,|\Lambda_-|\sigma\}|\Lambda_-|\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|\Lambda_-|)e^{\Lambda_-\tau}\int_1^{\frac{t+1}{2}} |\Lambda_-|e^{|\Lambda_-|\sigma}|\Lambda_-|\sigma\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right)ds \\
&+ \text{const. } (1+|\Lambda_-|)e^{\Lambda_-\tau}\int_{\frac{t+1}{2}}^t |\Lambda_-|e^{|\Lambda_-|\sigma}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|\Lambda_-|)\left(\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^2}\tilde{\mu}_\alpha\right) \quad ,
\end{aligned}
$$

which shows that $\kappa\partial_k\hat{\omega}_{3,1,1}\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},\frac{3}{2}}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{3,2,1}$ uses (74), Proposition 18 and (69), leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{3,2,1}| &= \left|\frac{1}{2}e^{-\kappa\tau}\int_t^\infty \check{f}_{2,1}(k,s-1)\kappa\partial_k\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } e^{\Lambda_-\tau}\int_t^\infty (1+|k|)|\Lambda_-|e^{-|k|\sigma}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|k|)e^{\Lambda_-\tau}e^{|k|\tau}\int_t^\infty (|k|^{\frac{1}{2}}+|k|)e^{-|k|\sigma}\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const. } (1+|k|)e^{\Lambda_-\tau}\left(\frac{1}{t^1}\bar{\mu}_\alpha+\frac{1}{t^{\frac{3}{2}}}\tilde{\mu}_\alpha\right) \leq \text{const. } (1+|k|)\frac{1}{t^1}\tilde{\mu}_\alpha\ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{3,2,1}\in\mathcal{D}^1_{\alpha-1,\infty,1}$.

The bound on the function $\kappa\partial_k\hat{\omega}_{3,3,1}$ uses (75) and Proposition 17, leading to

$$\begin{aligned}
|\kappa\partial_k\hat{\omega}_{3,3,1}| &= \left|\frac{1}{2}(e^{\kappa\tau}-e^{-\kappa\tau})\int_t^\infty \check{f}_{3,1}\kappa\partial_k\hat{Q}_1(k,s)ds\right| \\
&\leq \text{const. } (e^{|\Lambda_-|\tau}+e^{\Lambda_-\tau})\int_t^\infty e^{\Lambda_-\sigma}\,|\Lambda_-|\left(\frac{1}{s^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{s^2}\tilde{\mu}_\alpha\right)ds \\
&\leq \text{const.}\left(\frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha+\frac{1}{t^2}\tilde{\mu}_\alpha\right)\ ,
\end{aligned}$$

which shows that $\kappa\partial_k\hat{\omega}_{3,3,1}\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},2}$.

Collecting the bounds we have that $\hat{d}_3\in\mathcal{D}^1_{\alpha-1,\frac{3}{2},1}\subset\mathcal{D}^1_{\alpha-1,\frac{3}{2},0}$, which proves Lemma 8.

A Convolution with the semi-groups $e^{\Lambda_- t}$ and $e^{-|k|t}$

To make this paper self-contained, we recall the following results proved in [12]. In order to bound the integrals over the interval $[1,t]$ we systematically split them into integrals over $[1,\frac{t+1}{2}]$ and integrals over $[\frac{t+1}{2},t]$ and bound the resulting terms separately. For the semi-group $e^{\Lambda_- t}$ we have:

Proposition 15 *Let $\alpha\geq 0$, $r\geq 0$ and $\delta\geq 0$ and $\gamma+1\geq\beta\geq 0$. Then,*

$$e^{\Lambda_-(t-1)}\int_1^{\frac{t+1}{2}} e^{|\Lambda_-|(s-1)}|\Lambda_-|^\beta\frac{(s-1)^\gamma}{s^\delta}\mu_{\alpha,r}(k,s)\ ds$$
$$\leq\begin{cases}\text{const. }\dfrac{1}{t^\beta}\tilde{\mu}_\alpha(k,t), & \textit{if } \delta>\gamma+1\\[2ex] \text{const. }\dfrac{\log(1+t)}{t^\beta}\tilde{\mu}_\alpha(k,t), & \textit{if } \delta=\gamma+1\\[2ex] \text{const. }\dfrac{t^{\gamma+1-\delta}}{t^\beta}\tilde{\mu}_\alpha(k,t), & \textit{if } \delta<\gamma+1\end{cases}$$

uniformly in $t\geq 1$ and $k\in\mathbb{R}$.

Proposition 16 *Let $\alpha\geq 0$, $r\geq 0$, $\delta\in\mathbb{R}$, and $\beta\in\{0,1\}$. Then,*

$$e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^t e^{|\Lambda_-|(s-1)}|\Lambda_-|^\beta\frac{1}{s^\delta}\mu_{\alpha,r}(k,s)\ ds\leq\frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,r}(k,t)\ ,$$

uniformly in $t\geq 1$ and $k\in\mathbb{R}$.

For the integral over the interval $[t,\infty)$ we need only one of the bounds in [12].

Proposition 17 *Let $\alpha \geq 0$, $r \geq 0$, $\delta > 1$, and $\beta \in \{0,1\}$. Then,*

$$e^{|\Lambda_-|(t-1)} \int_t^\infty e^{\Lambda_-(s-1)} |\Lambda_-|^\beta \frac{1}{s^\delta} \mu_{\alpha,r}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(k,t)\ ,$$

uniformly in $t \geq 1$ and $k \in \mathbb{R}$.

For the semi-group $e^{-|k|t}$ we have:

Proposition 18 *Let $\alpha \geq 0$, $r \geq 0$, $\delta > 1$, $\beta \in [0,1]$ Then,*

$$e^{|k|(t-1)} \int_t^\infty e^{-|k|(s-1)} |k|^\beta \frac{1}{s^\delta} \mu_{\alpha,r}(k,s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(k,t)\ ,$$

uniformly in $t \geq 1$ and $k \in \mathbb{R}$.

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